Universality in Two Dimensional Gauge Theory

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We discuss two dimensional Yang – Mills theories with massless fermions in arbitrary representations of a gauge group G. It is shown that the physics (spectrum and interactions) of the massive states in such models is independent of the detailed structure of the model, and only depends on the gauge group G and an integer k measuring the total anomaly. The massless physics, which does depend on the details of the model, decouples (almost) completely from that of the massive one. As an example, we discuss the equivalence of QCD_2 coupled to fermions in the adjoint, and fundamental representations.

1. Introduction

It is well known [1-3] that many conformal field theories (CFT) invariant under a global symmetry based on a Lie algebra G actually possess an infinite affine Lie algebra Kac Moody (KM) symmetry $\widehat{G}$, generated by conserved currents $J^a(z) = \sum_n J_n^a z^{-n-1}$, satisfying the operator product expansion / commutation relations:

$$
\begin{aligned}
J^a(z)J^b(w) &= \frac{k\delta^{ab}}{(z-w)^2} + if^{abc}\frac{J^c(w)}{z-w}, \\
[J_n^a, J_m^b] &= kn\delta_{n+m,0}\delta^{a,b} + if^{abc}J_{n+m}^c.
\end{aligned}
\tag{1.1}
$$

The integer k is the level of the KM algebra, f^{abc} the structure constants of G. The Hilbert space of such CFT's can be described as an in general infinite direct sum of highest weight representations which are obtained by acting with current creation operators $J_{n<0}^a$ on primaries of the KM algebra $\widehat{G}$. CFT's can be classified by G, k, and by the list of highest weight representations appearing in a given model.

It is natural to ask what happens when such systems are coupled to non – Abelian gauge fields, i.e. when one considers the quantum theory based on the Lagrangian:

$$
\mathcal{L} = \mathcal{L}_{CFT} + \bar{A}J + A\bar{J} + \frac{1}{g^2}F^2,
\tag{1.2}
$$

where $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}]$. The main purpose of this note is to show that the massive physics in such models is actually *independent* of the particular realization of $\widehat{G}_k$ in the CFT. The massless sector described by a coset model [4], which carries the information about the particular realization of $\widehat{G}$, decouples almost completely from the massive physics.

It is useful to illustrate some of the issues that arise in a solvable example, the generalized Schwinger model: a set of n right moving complex fermions ψ_i with $U(1)$ charges q_i $(i = 1\cdots n)$, and m left moving fermions $\bar{\chi}_j$ with charges p_j $(j = 1\cdots m)$, coupled to a $U(1)$ gauge field as in (1.2). This system was studied in [5] where it was shown that if the chiral anomaly vanishes, i.e.

$$
k_l \equiv \sum_{i=1}^n q_i^2 = \sum_{j=1}^m p_j^2 \equiv k_r,
\tag{1.3}
$$

the massive spectrum contains a single free scalar field with mass $M^2 = g^2k$ $(k = k_l = k_r)$. The mass M is independent of q_i, p_j and n, m. In addition there are some massless particles (see below).

To reveal the KM structure it is convenient to study the model in a chiral gauge, $A^z \equiv A = 0$. In this gauge the action (1.2) takes the form:

$$\mathcal{L} = \psi_i^\dagger \bar\partial \psi_i + \bar\chi_j^\dagger \partial \bar\chi_j + \bar{A}J + \frac{1}{g^2}(\partial\bar{A})^2, \qquad (1.4)$$

where $J = \sum q_i \psi_i^\dagger \psi_i$. The "Gauss Law" constraint $\delta\mathcal{L}/\delta A = 0$ enforces current conservation $\partial\bar{J} + \bar\partial J = 0$. Integrating out $\bar{A}$ leads to

$$\mathcal{L} = \mathcal{L}_{CFT} + g^2 J \frac{1}{\partial^2} J. \qquad (1.5)$$

Bosonizing ψ_i, $\bar\chi_j$ we find n right moving chiral scalars H_i and m left moving ones $\bar{H}_j$. The interaction in (1.5) gives mass to one combination of the H_i, $\sqrt{k}H = \sum q_i H_i$, with $i\sqrt{k}\partial H = J$ (H, H_i are canonically normalized). Of course, a massive particle may no longer be chiral. Current conservation (and the related anomaly cancellation condition (1.3)) relates one of the m left moving scalars $\sqrt{k}\bar{H} = \sum p_j \bar{H}_j$ to H: $\bar{H} = H$. Thus the gauge theory (1.4) describes one free massive and $n - 1$ ($m - 1$) free massless right (left) moving particles. The flavour symmetry ($U(l)$ for charges appearing l times) acts on the massless sector, which is decoupled from the massive state $H(= \bar{H})$. We will see below that in non abelian gauge theories the massive and massless sectors will in general be separately strongly interacting, but their mutual decoupling is in fact general. As here, it is only the massless sector (a coset CFT) that will carry the information regarding the set of representations of G one started with (here q_i, p_j, n, m), and possible flavour symmetries.

To discuss the KM structure of massless gauge theories, it is further convenient to employ Hamiltonian light – front quantization of (1.4), (1.5) [6,7]. One treats $x^- = z$ as a spatial coordinate, and $x^+ = \bar{z}$ as "time." Canonical quantization of (1.5) then leads to a Hilbert space spanned by creation operators of ψ_i satisfying the standard anticommutation relations[1], with

$$P^+ = \int dz \psi^\dagger \partial\psi,$$
$$P^- = \int dz (\frac{1}{\partial}J)^2. \qquad (1.6)$$

The light – cone momentum operator P^+ is diagonal on the ψ Hilbert space; one is looking for eigenstates of the light – cone Hamiltonian P^-. We see that it is natural to describe

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[1] It is easy to see that the light – cone gauge Gauss law mentioned above does not lead to constraints on this Hilbert space.

the dynamics in terms of the KM structure, since the Hamiltonian acts inside current blocks. For a $U(1)$ gauge group, (1.4), (1.5), current blocks are labeled by the charge of the primary J_0. In the gauge theory we have to impose the condition

$$J_0|\text{phys}\rangle = 0 \tag{1.7}$$

arising from fixing the residual $\bar{z}$ dependent gauge symmetry $\delta\bar{A} = \bar{\partial}\epsilon(\bar{z})$ of (1.4). There-fore, physical states must all come from the current block of the identity. In particular, it is easy to verify using (1.1), (1.6) that the state $|p^+\rangle = J(-p^+)|0\rangle$ satisfies: $P^+|p^+\rangle = p^+|p^+\rangle$, $P^-|p^+\rangle = \frac{g^2 k}{p^+}|p^+\rangle$, corresponding to the massive state mentioned above as seen in the in-finite momentum frame.

Note that the Hilbert space, which is the Fock space of ψ_i, does not include the left moving degrees of freedom $\bar{\chi}$. Mathematically, surfaces with $x^+ = $ const do not define a Cauchy problem for $\bar{\chi}$. Physically, light – front quantization (1.6) describes the Hilbert space seen by an observer moving with the speed of light to the right. Such an observer can see all massive particles, as well as all right moving massless particles, but misses massless left moving ones.

It is important that while one misses the $m - 1$ massless left moving scalars, one *does not* miss any massive bound states of these massless constituents. From the point of view of a stationary frame, the point is that massive bound states involve non – trivial coupling between left and right moving constituents, which in general in massless $2d$ gauge theory occurs only through the anomaly, i.e. only through the current sector[2]. It is easy to show that the contact interaction $\langle J(z)\bar{J}(0)\rangle = \pi\delta^2(z)$ in the free CFT ensures current conservation in (1.4). Thus, while in light – front quantization the physical Hilbert space (the ψ's) is only "half" of the full space of fixed time quantization (which also includes the χ's), for *massive* physics the only aspect of the $\bar{\chi}$'s that enters is the current $\bar{J}$, which can be expressed in terms of J, $\bar{J} = -\frac{1}{\partial}\bar{\partial}J$. Hence, the ψ Hilbert space of light – front quantization is sufficient to discuss massive physics (as well as physics of the massless sector of ψ). In fact light – cone gauge together with quantization in the infinite momentum frame makes (in two dimensions) the decoupling between massive and massless, and left and right moving particles most manifest.

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[2] Equivalently, in terms of space – time propagation, the only way right moving massless particles can turn into left moving ones is through pair annihilation into photons.

Another important point that should be mentioned here concerns infrared regularization. This is usually achieved by studying the physical system on a spatial circle of finite radius. The space – time picture of bound state formation mentioned above implies that such a regularization is *inappropriate* for our purposes, since it destroys the decoupling between left and right movers. In addition to the coupling through the current sector, a right moving particle can then interact with a left moving one by going around the circle[3]. Light front quantization suggests a natural alternative: one can treat "space", x^- as compact, keeping "time", x^+ non – compact. This respects the decoupling between left and right movers, and we will use it below.

The arguments above seem general, so in the next section we will attempt to generalize them to the case of non – abelian gauge theories where they can be used to study the relation between much less trivial theories.

2. The decoupling theorem.

Our previous comments suggest a strategy of dealing with two dimensional Yang – Mills theories with massless quarks in arbitrary representations of the gauge group. The purpose is to show that the physics of massive bound states in such theories depends only on the gauge group G and the KM level k (see (1.1)), and is independent of the detailed representation content of the theory.

Consider the gauge theory Lagrangian (1.2) with right handed quarks $\psi^{(r)}$ and left handed ones $\bar{\chi}^{(r')}$, in representations r and r' of G respectively. Thus:

$$\mathcal{L}_{CFT} = \sum_r \psi^{\dagger(r)}\bar{\partial}\psi^{(r)} + \sum_{r'} \bar{\chi}^{\dagger(r')}\partial\bar{\chi}^{(r')} \tag{2.1}$$

with the KM currents (1.1) given by:

$$\begin{aligned} J^a &= \sum_r \psi^{\dagger(r)}\lambda^{a(r)}\psi^{(r)} \\ \bar{J}^a &= \sum_{r'} \bar{\chi}^{\dagger(r')}\lambda^{a(r')}\bar{\chi}^{(r')}, \end{aligned} \tag{2.2}$$

where $\lambda^{a(r)}$ are G matrices in the representation r, and summation over gauge indices has been suppressed.

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[3] Of course, as the radius of the circle goes to infinity this coupling dissappears.

The gauge theory (1.2) is believed to be consistent if the chiral anomaly vanishes, i.e. if the levels $k, \bar{k}$ of the right and left KM algebras (2.2) coincide, $k = \bar{k}$. Proceeding as in the Schwinger model, we choose the light – cone gauge $A^z = 0$, and perform light – front quantization with $z =$ space, $\bar{z} =$ time. As discussed above, this makes the decoupling of left and right movers most transparent. The Hilbert space of the light – front theory, which consists of ψ creation operators satisfying canonical anticommutation relations acting on the vacuum, is just that of the right moving ψ CFT (2.1).

The Gauss law obtained by varying $\mathcal{L}$ (1.2) w.r.t. A^z enforces again current conservation in the quantum theory,

$$\bar{J} = -\frac{1}{\partial} \bar{D} J \tag{2.3}$$

and contains no new information, simply stating the form of $\bar{J}$ on the ψ Hilbert space. This complete decoupling of $\bar{\chi}$ is due (as before) to the fact that in the frame moving to the right with the speed of light, one can not see the left moving massless particles.

The form of the light – front Hamiltonian (1.6) suggests splitting the Hilbert space into $\widehat{G}$ current blocks. Putting $x^- = z$ on a circle, one finds:

$$P^- = \sum_{n=1}^{\infty} \frac{1}{n^2} J^a_{-n} J^a_n. \tag{2.4}$$

Thus, P^- acts inside current blocks as before, and the problem of finding the (massive) spectrum splits into decoupled CFT diagonalization problems for the operator P^- (2.4) on global G singlets (as in (1.7)) in the different current blocks.

What's most important for our purposes is that this description of the light – front dynamics is completely insensitive to the properties of the left moving sector $\bar{\chi}$ (1.2), (2.1). The only feature of $\bar{\chi}$ that has been used is anomaly cancellation, $k = \bar{k}$. In particular, nothing prevents us from replacing $\bar{\chi}$ by another massless CFT with the same $\bar{k}$. Gauge and Lorentz invariance of the chiral theory (1.2), (2.1) allow us now to study the system in the two different light – cone gauges in appropriate infinite momentum frames and conclude that ψ too can be replaced by another set of fermions with the same k without changing the massive spectrum. Since the sets of representations r, r' are arbitrary, we arrive at our main result: the spectrum depends only on the gauge group G and the KM level (or total anomaly) k. It does not depend on the detailed list of representations r leading to that total anomaly k. In particular, the spectra of the non chiral (left – right symmetric) theories corresponding to ψ, χ should coincide.

The essential ingredients of the above argument are decoupling of left movers in a right moving infinite momentum frame and the fact that the spectrum of massive particles that can be boosted to $v = +c$, which can be thought of as bound states of ψ, J should be the same as that of particles that can be boosted to $v = -c$ and can be thought of as bound states of $\bar{\chi}, \bar{J}$.

A few comments about this result are in order:

1) In principle one should be able to identify different states constructed from ψ, J in $A = 0$ gauge with $x^+ (= \bar{z}) = $ time with different states in $\bar{A} = 0$ gauge with $x^- = $ time. However the argument described above does not contain any detailed information about this mapping in general. This is an interesting open problem. We will study it in an example in section 3.

2) "Massive physics" above includes the spectrum of single particle states and their scattering amplitudes.

3) As explained in the introduction, some massless states are not seen in light – front quantization since the observer is moving with the speed of light. Massless states can be studied by familiar coset CFT techniques which we discuss in section 4.

4) For our results to be true, there must exist a complete decoupling between massive physics, which is universal, and massless physics as well as topological effects, which are model dependent. This is indeed the case in two dimensional gauge theory up to some global correlations to be discussed below. As an example, flavour symmetry, which clearly depends on the representation content can not be universal; consequently it must be carried by the massless sector.

5) Because of the above decoupling, physics of the massive states may exhibit symmetries that are not apparent in the Lagrangian. E.g. despite the fact that (1.2), (2.1) is not in general parity invariant, the parity violation is carried by the massless sector.

6) Due to the algebraic nature of the light – front Hamiltonian P^- (2.4) it is clear that in all current blocks that are shared by two different theories based on a given KM symmetry $\widehat{G}_k$ there is an independent argument that physics will indeed be the same. Thus, for example any two such gauge theories share the identity current block, so at least part of the spectrum must be the same. The issue, from this point of view, is why the result is so general, when different theories with $\widehat{G}_k$ symmetry have in general distinct lists of KM primaries. To understand how this may happen, we discuss in the next section, a non – trivial example, $\widehat{SU(N)}_N$ in the limit $N \to \infty$.

3. An example: adjoint versus fundamental fermions in QCD$_2$.

There has been some recent interest in the dynamics of two dimensional gauge fields coupled to fermions in the adjoint representation of $G = SU(N)$ in the large N limit [8-11]. The model exhibits a rich spectrum of bound states with an infinite number of Regge trajectories and may be a useful toy model for large N QCD$_4$ as well as QCD strings. The level of the $\widehat{SU(N)}$ KM symmetry (1.1) generated by $J^{ab} = \psi^{ac}\psi^{cb}$, $a, b = 1 \cdots N$ is N. It is interesting to apply our analysis to this system, since few exact results about it are available. In particular, as a non – trivial check on our results one can compare adjoint QCD$_2$ to a model of complex fermions $\psi^{\alpha a}$, with a color index $a = 1 \cdots N_c = N$, and a flavour one $\alpha = 1 \cdots N_f = N$ (N flavours of fermions in the fundamental of $SU(N_c)$). The $SU(N_c)$ current $J^{ab} = \psi^{\dagger a\alpha}\psi^{\alpha b}$ generates a $\widehat{SU(N)}_N$ KM algebra as well. The arguments of the previous section would suggest that the two left – right symmetric theories have the same massive spectrum for all N, and in particular as $N \to \infty$ where the analysis simplifies. We will now look at these spectra in some detail and attempt to compare them.

3.1. Adjoint fermions.

It is convenient to put "space" $x^- = z$ on a circle as described above, and take the fermions ψ^{ab} to be antiperiodic (Neveu – Schwarz) around the circle. The Hilbert space of global $SU(N)$ singlets (1.7) is then spanned by states of the form:

$$\frac{1}{N^{l/2}} \mathrm{Tr}\left(\psi_{-r_1}\psi_{-r_2}\cdots\psi_{-r_l}\right)|0\rangle; \ \ 0 < r_i \in Z + \frac{1}{2}; \ \ l \geq 2, \tag{3.1}$$

where we take states with a single trace in (3.1) because of the large N limit. For massless constituents, the form of P^- (2.4) suggests arranging the Hilbert space in a different way, according to blocks of $\widehat{SU(N)}$ KM. The diagonalization of P^- splits into decoupled problems for the different current blocks. To specify the current blocks that appear, we need to determine the KM primaries in the model. The two simplest current blocks are:

1) The current block of the identity, with global $SU(N)$ singlets of the form

$$\frac{1}{N^l} \mathrm{Tr}\left(J_{-n_1}J_{-n_2}\cdots J_{-n_l}\right)|0\rangle; \ \ 0 < n_i \in Z; \ l \geq 2. \tag{3.2}$$

2) The adjoint current block, with global $SU(N)$ singlets:

$$\frac{1}{N^{l+\frac{1}{2}}} \mathrm{Tr}\left(J_{-n_1}J_{-n_2}\cdots J_{-n_l}\psi_{-\frac{1}{2}}\right)|0\rangle; \ \ 0 < n_i \in Z; \ l \geq 1. \tag{3.3}$$

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More complicated highest weight states appear in products of ψ's. Explicitly, in the space of states of the form[4]:

$$\prod_{i=1}^{n} \psi_{-\frac{1}{2}}^{a_i b_i} |0\rangle \tag{3.4}$$

one can find all representations of the form $T = \bar{S}R$ where the Young tableaux of S, R contain n boxes, and T is defined as follows: if R, S have columns of length c_i, $\tilde{c}_j$, $i = 1, \cdots, L$, $j = 1, \cdots, \tilde{L}$, then the Young tableau for T has columns of length

$$\begin{cases} N - \tilde{c}_{\tilde{L}+1-i} & i = 1, \cdots, \tilde{L} \\ c_{i-\tilde{L}} & i = \tilde{L}+1, \cdots, \tilde{L}+L \end{cases} \tag{3.5}$$

Furthermore, due to antisymmetry of (3.4) under interchange of any two fermions it is clear that the Young tableau for S has to be the transpose of that for R: $S = R^t$. Thus the highest weight representations of $\widehat{SU(N)}$ that appear in this model are all representations of the form $\bar{R}^t R$ with arbitrary R, with the length of the first row n_1 and first column c_1 in the Young tableau of R satisfying $n_1 + c_1 \leq N$ (the unitarity constraint of KM representation theory). A given R corresponds to a particular way of symmetrizing the indices a_i in (3.4). Each such highest weight state gives rise to a current block which contains global $SU(N)$ singlets, since all the representations involved are invariant under the center of $SU(N)$, Z_N; these can be written analogously to (3.2), (3.3). One can diagonalize P^- (2.4) separately on the different current blocks.

In the large N limit certain simplifications occur. All representations with given n in (3.4) collapse to one, as far as single particle states are concerned. The space of potential single particle states with given n is (schematically):

$$\text{Tr}\left(J^{l_1} \psi J^{l_2} \psi \cdots J^{l_m} \psi\right) |0\rangle. \tag{3.6}$$

To illustrate this, consider the case $n = 2$. The state

$$|\theta^{abcd}\rangle = \left(\psi_{-\frac{1}{2}}^{ab} \psi_{-\frac{1}{2}}^{cd} - \frac{1}{N}\delta^{bc} J_{-1}^{ad} + \frac{1}{N}\delta^{ad} J_{-1}^{cb} + (a \leftrightarrow c)\right) |0\rangle \tag{3.7}$$

is a primary (compare to (3.4) and the discussion following it), with R being the two index symmetric representation. A simple global $SU(N)$ primary is

$$J_{-n_1}^{da} J_{-n_2}^{bc} |\theta^{abcd}\rangle. \tag{3.8}$$

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[4] At finite N, it is necessary to project out of these states various contributions corresponding to descendants.

The terms in (3.8) have the form $\text{Tr}(J\psi J\psi)|0\rangle$, $(\text{Tr}J\psi)(\text{Tr}J\psi)|0\rangle$, and $\frac{1}{N}\text{Tr}(JJJ)|0\rangle$. Only the first of these can be in the single particle Hilbert space as $N \to \infty$. The second term corresponds to two particles and the third is down by $1/N$. The second representation with $n = 2$ in (3.4) corresponds to R = antisymmetric tensor; it differs from (3.7) by some signs, corresponding to antisymmetrization rather than symmetrization in a, c. At large N these signs control only the relative weight of the one and two trace terms in (3.7). In general, we see that different representations $\bar{R}^t R$ where R has n boxes differ by their projections on different multiparticle states. For given R the states are particular linear combinations of $1, 2, \cdots, n$ trace states (3.6). If one can identify single trace states with single particle states, the single particle spectrum should split into sectors labeled by $n = 0, 1, 2, \cdots$ (3.6), in each of which the Hamiltonian acts independently. We will soon see that actually only the sectors with $n = 0, 1$ (i.e. (3.2), (3.3)) give rise to single particle states.

3.2. N flavours of fundamental fermions.

Only representations invariant under the center of $SU(N)$ (which in this case are a subclass of all existing representations) should be considered. In the baryon number zero sector, we have states of the form (compare to (3.4)):

$$\prod_{i=1}^{n} \psi^{\dagger a_i \alpha_i}_{-\frac{1}{2}} \psi^{\beta_i b_i}_{-\frac{1}{2}} |0\rangle. \tag{3.9}$$

It is easy to repeat the previous analysis for this case. Since the flavour indices are at our disposal, we can separately symmetrize a_i, b_i in an arbitrary fashion. Hence, we find for given n all highest weight states in representations of the form $\bar{S}R$ with R, S corresponding to arbitrary Young tableaux with n boxes. Of course, the (anti-) symmetry of (3.9) implies that given R, S the transformation properties under the flavour group are determined to be $\bar{S}^t R^t$. Thus, we see that the list of representations of $SU(N_c)$ here is larger than the one obtained in the adjoint case.

At large N there is again a significant simplification. Multiplying (3.9) by products of currents to form global singlets we see that all representations $\bar{S}R$ with R, S containing n boxes give linear combinations of states of the form:

$$\left[\psi^{\alpha_1 a_1}\left(J^{l_1}\right)^{a_1 b_1}\psi^{\dagger b_1 \beta_1}\right]\left[\psi^{\alpha_2 a_2}\left(J^{l_2}\right)^{a_2 b_2}\psi^{\dagger b_2 \beta_2}\right]\cdots|0\rangle. \tag{3.10}$$

We see that all such representations with given $n \geq 1$ give rise to different n particle states. But since the diagonalization of P^- (2.4) is an algebraic problem insensitive to

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the realization of the different current blocks, the conclusion must be valid in any other representation of $\widehat{SU(N)}_N$ as well. This implies that for $\widehat{SU(N)}_{N\to\infty}$, the only current blocks contributing single particle massive states are the identity block (3.2), and the adjoint block (3.3). Both appear in the two theories we are comparing, the identity trivially (3.2) and the adjoint by replacing $\psi^{ab} \to \psi^{\dagger a\alpha}\psi^{\beta b}$ in (3.3). The higher representations, which as we saw are different in the two cases, contribute only multiparticle states, and so the difference is unimportant.

3.3. Comments.

One interesting property of the mapping of states described above between the adjoint and fundamental theories is that the fermionic states in the adjoint current block in (3.3) are mapped into bosonic ones in the N flavour fundamental theory. The reason for the discrepancy is clear: while a state like $\mathrm{Tr}(J^n\psi)|0\rangle$ in the adjoint theory is a (fermionic) single particle state, in the fundamental theory it corresponds to $(J^n)^{ab}\psi^{\dagger b\alpha}\psi^{\beta a}|0\rangle$ which has non trivial flavour content. Since P^- (2.4) is completely insensitive to flavour, such states correspond to two particle states where a massive particle interacting with the color field is accompanied by a massless state sensitive only to flavour. The two particle state is bosonic but it is difficult in general to determine (independently) the statistics of the massive state alone, since there is no state in the Hilbert space of the theory with only the massive component (see section 4 for more detail). This pecularity is of no dynamical significance since, as mentioned above, the massless state accompanying the massive one is a spectator that does not participate in the dynamics.

The same can be said in general about the multiplicities and flavour content of various states. Counting of states requires taking into account the decoupled massless flavour degrees of freedom. After doing that, the massive dynamics is found to be the same.

Note that we have found that many sectors that naively contain single particle states in the large N limit in adjoint QCD$_2$ ((3.6) with $n > 1$) actually give rise to multi particle states. While it is possible to show this directly in the adjoint model, it is more apparent in the fundamental representation (3.10).

Finally, we haven't discussed here the non zero baryon number sectors[5]. At large N, baryons become heavy (the ones that are exactly massless because they belong to the

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[5] $U(1)_B$, like flavour is carried only by the massless sector. Still these sectors give rise (in general) to new $SU(N_c)$ highest weight representations.

decoupled coset CFT are of no interest to us), and therefore are outside the scope of the present analysis. At finite N there should presumably be a mapping of the two models that includes them. It is important to reiterate that while the analysis in this section relied on certain simplifications of the large N limit of the gauge theories, the equivalence of the massive physics should be a property of these theories for all values of N.

4. The massless sector

In this section we'll study in more detail the (de)coupling between the massless and massive sectors of $2d$ gauge theories. The exact structure of the massless sector of the theory (1.2) is most transparent in the A_0 gauge. We use the Schrodinger representation, the left and right moving fermions $\psi(x)$ and $\bar{\chi}(x)$ being operators on a circle $0 \le x \le 2\pi$ with antiperiodic boundary conditions. The gauge invariant Hilbert space is defined by projecting to the 0 – eigenvalue states of the Gauss operator $I(x)$:

$$I(x) = J_L(x) + J_R(x) - [D_x E(x)]^a, \tag{4.1}$$

where $J_L^a(x) = \sum_r \psi^{\dagger(r)} \lambda^{a(r)} \psi^{(r)}$, $J_R^a(x) = \sum_{r'} \bar{\chi}^{\dagger(r')} \lambda^{a(r')} \bar{\chi}^{r'}$, $E(x)$ is the chromoelectric field , D_x is the covariant derivative and r and r' run over the set of left and right fermionic representations, respectively.

The Hilbert space of the fermions can be decomposed using the coset construction [4]:

$$\mathcal{H}_L = \sum_s \oplus (\mathcal{H}_s^c \otimes \mathcal{H}_s^{\widehat{G},k}) \quad \mathcal{H}_R = \sum_{s'} \oplus (\mathcal{H}_{s'}^c \otimes \mathcal{H}_{s'}^{\widehat{G},k}) \tag{4.2}$$

where $\mathcal{H}_s^{\widehat{G},k}$ is a highest weight s representation of the KM algebra $\widehat{G}$ at level k, and $\mathcal{H}_s^c$ are blocks of the appropriate coset theory. The only property of the cosets we will need in the following is that they accomodate an action of the Virasoro algebra, i.e. they correspond to sets of massless representations of the Lorentz group. Generally these representations appear in nontrivial superpositions which do not admit a simple interpretation in terms of massless fermions or bosons.

From (4.1) it is clear that the projection to 0 – eigenvalues involves only the representations of $\widehat{G}$, i.e. for fixed s, s' we should find the states in the product space $\mathcal{H}_s^{\widehat{G},k} \otimes \mathcal{H}_{s'}^{\widehat{G},k} \otimes \mathcal{H}^A = \mathcal{H}^P$ which are annihilated by $I(x)$ (4.1), where $\mathcal{H}^A$ is the Hilbert space spanned by the eigenstates of the gauge potential $A^a(x)$. This defines the gauge

invariant Hilbert space $\mathcal{H}^{GI}$. Since the gauge potential is invariant under the center of G, solutions of (4.1) exist only if s and s' belong to the same element of the center.

The hamiltonian in the $A_0 = 0$ gauge is :

$$
\begin{aligned}
H = &\frac{1}{k+h} \int_0^{2\pi} \left[: J_L^2(x) + J_R^2(x) : -2kA(x)(J_L(x) - J_R(x)) + 2A(x)^2 \right] dx \\
&+ \frac{g^2}{2} \int_0^{2\pi} (E(x))^2 dx + L_0^c + \bar{L}_0^c.
\end{aligned}
\tag{4.3}
$$

$L_0^c, \bar{L}_0^c$ are the Virasoro generators acting on the coset, and h is the dual Coxeter number of G. The hamiltonian decomposes into a part acting on $\mathcal{H}^{GI}$ and a decoupled piece acting on the coset. The states in the left and right cosets remain therefore in the gauge invariant Hilbert space and represent massless, decoupled degrees of freedom.

Before studying their properties further we have to make sure that there are no "accidentally massless" states produced by the diagonalization of the $\mathcal{H}^{GI}$ (coupled) part. Since the coupled theory is superrenormalizable, the infrared limit is obtained when the coupling flows to infinity. In this limit the gauge field kinetic term disappears and the coupled part of the theory becomes a $\widehat{G}/\widehat{G}$ topological theory, i.e. there are no massless degrees of freedom left.

To complete the discussion of the physical Hilbert space we need to consider the role of "big" gauge transformations present when the left and right fermion representations are invariant under elements g_0 of the center of G [12]; e.g. for adjoint QCD$_2$ the gauge group is $SU(N)/Z_N$; big gauge transformations are labeled by elements of Z_N.

Consider a big gauge transformation $g(x)$ corresponding to one of these elements :

$$
g(x + 2\pi) = g(x)g_0
\tag{4.4}
$$

Such a transformation leaves the states of the coset inert and has a very simple action on $\mathcal{H}^P$: besides transforming $A^a(x)$ in the usual way, it induces an outer automorphism of the KM algebra and therefore takes a representation s into a representation s^{g_0}. The exact correspondence follows from the permutation of the weights induced by g_0 on the extended Dynkin diagram of $\widehat{G}$ [3]. The eigenstates of the big gauge transformations (i.e. "θ- vacua" and states built on them) are linear combinations:

$$
\sum_{g_0} exp(if(\theta, g_0)) \ \mathcal{H}_{s^{g_0}}^{\widehat{G},k} \otimes \mathcal{H}_{s'^{g_0}}^{\widehat{G},k} \otimes \mathcal{H}^{A^g}
\tag{4.5}
$$

where the phases $f(\theta, g_0)$ form a representation θ of the group composed by the g_0 elements.

One can ask whether the big gauge transformations (4.4) lead to non – trivial dynamical effects (vacuum mixing). The answer is that they do not have any observable consequences. The reason is that the hamiltonian (4.3) is diagonal in the representations of $\widehat{G}$; it has no off diagonal matrix elements between different terms in (4.5). Moreover diagonal elements between representations related by g_0 will be the same. It follows that physical quantities do not depend on θ. Therefore models having fermions with different behaviour under the center can have the same massive physics. Note that the θ-independence arises without the presence of a global symmetry in the lagrangian, unlike in four dimensions or in the Schwinger model.

To illustrate the above procedure, consider the case discussed in Section 3, i.e. an $SU(N)$ colour group coupled to fermions in the adjoint representation. In this case g_0 can be any of the elements of the center

$$g_0 = exp(i\frac{2\pi}{N}n) \qquad n = 0, 1, 2, \cdots, N-1. \tag{4.6}$$

The automorphism corresponding to (4.6) with $n = 1$, takes the highest weight representation characterized by the Young tableau $(n_1, n_2, ..., n_{N-1})$ to the representation corresponding to $(N - n_{N-1}, n_1 - n_{N-1}, ..., n_{N-2} - n_{N-1})$, where $n_i \quad i = 1, ..., N-1$ denote the length of the rows. In particular, by taking the R representations (in the notation of Section 3) defined by Young tableaux with $n_1 = k$, $n_2 = n_3 = ... = 0$ one generates all the representations which could appear.

In light – front quantization the big gauge transformations are realized in an amusing fashion: on a given representation one should simultaneously change the hamiltonian by the outer automorphism acting on the currents and require singlet states under the global charges corresponding to the transformed currents.

The general structure of the Hilbert space after diagonalizing the hamiltonian , is:

$$\sum_{s,s'} \oplus(\mathcal{H}_s^c \otimes \mathcal{H}_{s,s'}^{GI} \otimes \mathcal{H}_{s'}^c) \tag{4.7}$$

where $\mathcal{H}_s^c$ contain massless states and all the states in $\mathcal{H}_{s,s'}^{GI}$ are massive. From (4.7) it is clear that even though all the information about the massive states is in $\mathcal{H}_{s,s'}^{GI}$, there is no factorization in the mathematical sense between the massive and massless states: the asymptotic state involves generally some massless component. However since the massless

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states do not interact in the Virasoro basis, the S-matrix is completely determined by the massive part.

Generally the cosets $\mathcal{H}_s^c$ represent a conformal theory which cannot be simply described by free fermions or bosons. If there is a continuous global symmetry present, since it acts on Weyl fermions it is chirally conserved. As a consequence it produces a KM algebra which will be part of the chiral algebra of the coset. Therefore all the continuous flavour symmetries are realized on the massless states. The massive parts have charge zero under all the continuous symmetries, including baryon number. Discrete symmetries like fermion number parity, act on the massive states, however due to the structure (4.7) they have well defined values only when the massive states are accompanied by the massless spectators.

The decoupled massless states are completely determined by the coset construction (4.2) and they do not have any dynamics unlike in four dimensions. It could happen that two different theories would lead to identical cosets or to cosets having some blocks in common, i.e. to the same theory in the infrared. Moreover even when the cosets have an interpretation in terms of massless bosons or fermions, the field operators of this objects are usually nonpolynomial in the basic fields , behaving in this sense as "solitons". These facts have some superficial resemblance to recently discovered features of four dimensional supersymmetric QCD [13].

A simple example showing these features is provided by the following two theories: color group $G = SU(N-2)$ with N multiplets of complex fermions in the fundamental representation and color group $G = SU(N+1)$ with $N-1$ multiplets in the fundamental representation. The massless spectrum for both theories is given by blocks described by $\frac{N(N-1)}{2}$ complex fermions. The baryon number of the fermions is 2 in units where the quarks have baryon number 1, showing that they are solitons.

5. Discussion.

The main results obtained here are:

1) The physics of massive bound states in two dimensional gauge theory with massless quarks is independent of most of the details of the representations in which the quarks lie, and only depends on the gauge group G and the current algebra level k.

2) There is complete decoupling between the physics of the massive bound states, which is universal, and that of the massless sector, which carries all the information about flavour

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symmetries, baryon number, etc. In general, this decoupling is somewhat obscured by the fact that states in the Hilbert space have non – trivial projections on both the massless and the massive sectors, as is familiar from coset models. To exhibit the decoupling it is very useful to quantize the system on the light – front. The whole structure is special to two dimensions and does not seem to have a simple counterpart in $3+1$ dimensional QCD, where there are strong interactions between massless and massive particles.

3) The point of view presented here leads to possible significant improvements in numerical studies of QCD_2 with massless quarks. One learns that the dynamics splits into current blocks, and furthermore in the example studied in detail very few current blocks (two) lead to single particle states. The resulting picture is very useful for thinking about boson – fermion cancellation in adjoint QCD_2, which was studied in [10], [11] on the basis of a similar cancellation that is generic in string theory [14]. Here all bosons come from the identity sector (3.2), while all fermions come from the adjoint block (3.3); it is clear that the spectrum of highly excited states is going to be dominated by states with large l in (3.2), (3.3) and hence will be the same for bosons and fermions. It therefore is plausible that $\mathrm{Tr}(-)^F \exp(-\beta H)$ exhibits here the cancellations typical of string theory [14]. We also saw that many sectors of the Hilbert space that naively contribute single particle states, actually give rise to multi particle states in the large N limit. It is important to take this effect into account in numerical estimates of the density of highly excited states in adjoint QCD_2 [9].

We haven't succeeded in understanding the results regarding the above universality in fixed time quantization, or in the path integral formalism. Also, the chiral gauge theories (1.2) play a central role. It would be interesting to understand the results in different ways. Clearly, it would be interesting to solve for the universal behaviour of the massive sector of $2d$ gauge theories. The algebraic approach followed here as well as the use of different realizations of a given theory should prove useful for that.

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